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Automorphic Functions on Hecke Subgroups Associated with Generalized Modular Equation

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Abstract: In this paper, the Hecke subgroup denoted by K associated with the generalized modular equation is determined. The generators of K are

$$A = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix},$$

where $\delta = 2\cos\frac{(1-2s)\pi}{2}$ for $s \in \left(0, \frac{1}{2}\right]$. It is shown that the moduli α and β of the generalized modular equation are automorphic functions, respectively, on K and $P^{-1}KP$ for $P = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$.

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Introduction: For $n \in \mathbb{N} \cup \{0\}$ and $a, b, c \in \mathbb{C}$ $(c \neq 0, -1, -2, \ldots)$, the Gaussian hypergeometric function, ${}_2F_1(a, b; c; z)$, is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)\, n!} z^n, \quad |z| < 1,$$

where $(a, n) = 1$ if $n = 0$, $a \neq 0$ and $(a, n) = a(a+1)\cdots(a+n-1)$ if $n \geq 1$. For a detailed discussion, see [6, Chapter II] and [18, Chapter XIV]. For $s \in \left(0, \frac{1}{2}\right]$ and $n \in \mathbb{N} \setminus \{1\}$, the following equation is called the generalized modular equation (cf. [11]):

$$\frac{{}_2F_1(s, 1-s; 1; 1-\beta)}{{}_2F_1(s, 1-s; 1; \beta)} = n \frac{{}_2F_1(s, 1-s; 1; 1-\alpha)}{{}_2F_1(s, 1-s; 1; \alpha)}. \quad (1)$$

For simplicity, assume that

$$F(\alpha) = {}_2F_1(s, 1-s; 1; \alpha)$$

and we will use this notation in the rest of this paper. Many mathematicians studied the generalized modular equation (1) (cf. [1, 2, 4, 5, 11, 12]), especially, the prominent Indian mathematician S. Ramanujan broadly studied the equation (1) and find many identities involving the moduli α and β (cf. [7, 8, 9, 10]). In this paper, we will show that the moduli α and β are automorphic functions on some Hecke subgroups associated with the equation (1).

For the 2×2 identity matrix I_2, the projective special linear group $\mathrm{PSL}_2(\mathbb{R})$ is given by

$$\mathrm{PSL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1 \right\} / \{\pm I_2\}$$

(see [17, Chapter VII]). Let $\mathcal{H} = \{\alpha \in \mathbb{C} : \mathrm{Im}\, \alpha > 0\}$, then the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathcal{H}$ is given by

$$\alpha \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha = \frac{a\alpha + b}{c\alpha + d},$$

where $\alpha \in \mathcal{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$.

Let K be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Suppose $g: \mathcal{H} \to \mathbb{C}$ is a holomorphic function. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K$ and $\alpha \in \mathcal{H}$, if

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$$g\left(\frac{a\alpha+b}{c\alpha+d}\right)=(c\alpha+d)^k g(\alpha),$$

then g is known as a weight-k automorphic form. If $k=0$, then

$$g\left(\frac{a\alpha+b}{c\alpha+d}\right)=g(\alpha)$$

and g is called an automorphic function (cf. [13, 15]).

For an integer $m\geq 3$ and $\delta=2\cos\frac{\pi}{m}$, let

$$U=\begin{pmatrix}1 & \delta\\ 0 & 1\end{pmatrix} \text{ and } V=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Then the group $H=\langle U,V\rangle$ is known as the Hecke group. See [14] and [16] for details about the Hecke group. In this paper, we consider the subgroup K of H generated by

$$A=\begin{pmatrix}1 & \delta\\ 0 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix},$$

where $\delta=2\cos\frac{(1-2s)\pi}{2}$ for $s\in\left(0,\frac{1}{2}\right]$.
By using the following lemma, we will show that $K=\langle A,B\rangle$ is the Hecke subgroup associated with the generalized modular equation (1).

Lemma 1 ([3], Lemma 4.1). *For* $s\in\left(0,\frac{1}{2}\right]$, *if*

$$z(\alpha)=i\frac{F(1-\alpha)}{F(\alpha)},$$

then z *maps* $\mathcal{H}=\{\alpha\in\mathbb{C}:\operatorname{Im}\alpha>0\}$ *onto the following hyperbolic triangle*

$$D=\left\{z\in\mathcal{H}:0<Re\, z<\cos\frac{(1-2s)\pi}{2},\left|2z\,\cos\frac{(1-2s)\pi}{2}-1\right|>1\right\}$$

in the z-plane. At the vertices

$$z(1)=0, z(0)=\infty \qquad \textit{and} \qquad z(\infty)=e^{i\frac{(1-2s)\pi}{2}},$$

the interior angles of D *are* 0, 0 *and* $(1-2s)\pi$, *respectively.*

Main Results: In the following lemma, we find the appropriate group associated with the generalized modular equation (1).

Lemma 2. *The group associated with the generalized modular equation*

$$\frac{F(1-\beta)}{F(\beta)}=n\frac{F(1-\alpha)}{F(\alpha)}$$

is the Hecke subgroup $K=\langle A,B\rangle$, *where*

$$A=\begin{pmatrix}1 & \delta\\ 0 & 1\end{pmatrix} \textit{ and } B=\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}$$

for $\delta=2\cos\frac{(1-2s)\pi}{2}$, $s\in\left(0,\frac{1}{2}\right]$.

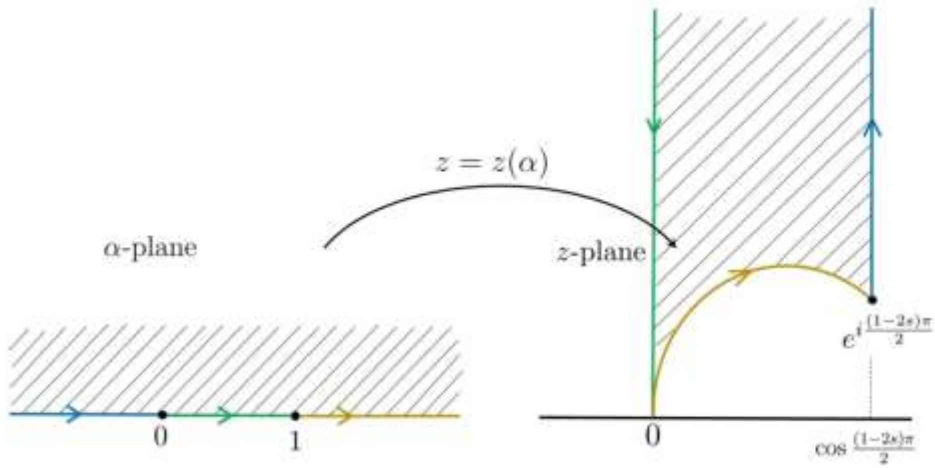

Fig. 1: The transformation $z(\alpha)$ maps $\mathcal{H}$ onto D.

Proof. For $s \in \left(0, \frac{1}{2}\right]$, let

$$z(\alpha) = i\frac{F(1-\alpha)}{F(\alpha)}.$$

By Lemma 1, $z(\alpha)$ maps $\mathcal{H} = \{\alpha \in \mathbb{C} : \text{Im}\ \alpha > 0\}$ onto

$$D = \left\{z \in \mathcal{H}: 0 < Re\ z < \cos\frac{(1-2s)\pi}{2}, \left|2z\ \cos\frac{(1-2s)\pi}{2} - 1\right| > 1\right\}$$

as shown in Fig. 1.

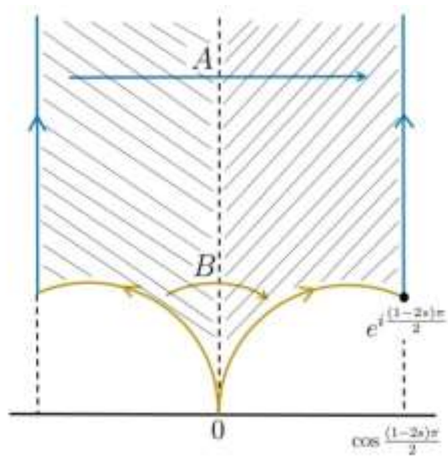

Fig. 2: Fundamental domain of $K = \langle A, B \rangle$.

If we reflect D about the y-axis, then the region shown in Fig. 2 serves as the fundamental domain of the group $K = \langle A, B \rangle$, where the side-pairing transformations A and B are the generators of the group K. Also, for

$$\delta = 2\cos\frac{(1-2s)\pi}{2}$$

and the Hecke group $H = \langle U, V \rangle$, where

$$U = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have

$$A = U = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = V^{-1}U^{-1}V = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}.$$

Hence we deduce that $K = \langle A, B \rangle$ is the required group. □

Lemma 3. *If*

$$z(\alpha) = i\frac{F(1-\alpha)}{F(\alpha)},$$

then

$$\alpha\left(-\frac{1}{z}\right) = 1 - \alpha(z).$$

Proof. Since

$$z(\alpha) = i\frac{F(1-\alpha)}{F(\alpha)},$$

we have

$$\begin{aligned} z(1-\alpha(z)) &= i\frac{F(\alpha)}{F(1-\alpha)} \\ &= -\frac{1}{z(\alpha)}. \end{aligned}$$

As α is the inverse of z, we have

$$1 - \alpha(z) = \alpha\left(-\frac{1}{z}\right).$$

□

In the following theorem, we prove by using Lemma 3 that $\alpha(z)$ and $\beta(z) = \alpha(nz)$ are automorphic functions on K and $P^{-1}KP$, respectively, where

$$P = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 1. *Let* $\delta = 2\cos\frac{(1-2s)\pi}{2}$ *for* $s \in \left(0, \frac{1}{2}\right]$. *If* $\alpha(z)$ *is the inverse of*

$$z(\alpha) = i\,\frac{F(1-\alpha)}{F(\alpha)}$$

and $\beta(z) = \alpha(nz)$, *then* $\alpha(z)$ *and* $\beta(z)$ *are automorphic functions, respectively, on* K *and* $K' = P^{-1}KP$, *where*

$$P = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

and K *is the group generated by*

$$A = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}.$$

Proof. If A' and B' are the generators of K', then

$$A' = P^{-1}AP = \begin{pmatrix} 1 & \frac{\delta}{n} \\ 0 & 1 \end{pmatrix}$$

and

$$B' = P^{-1}BP = \begin{pmatrix} 1 & 0 \\ n\delta & 1 \end{pmatrix}.$$

We need to show that

$$\alpha(Az) = \alpha(z), \quad \alpha(Bz) = \alpha(z)$$

and

$$\beta(A'z) = \beta(z), \quad \beta(B'z) = \beta(z).$$

Since $\exp\left(\frac{2\pi}{\delta}i(z+\delta)\right) = \exp\left(\frac{2\pi}{\delta}iz\right)$, we have $z + \delta = z$. Thus $\alpha(z+\delta) = \alpha(z)$, i.e., $\alpha(Az) = \alpha(z)$. Also,

$$\begin{aligned}\alpha(Bz) &= \alpha\left(\frac{z}{\delta z+1}\right)\\ &= \alpha\left(\frac{1}{\delta+\frac{1}{z}}\right).\end{aligned}$$

Now, using Lemma 3, we have

$$\begin{aligned}\alpha\left(\frac{1}{\delta+\frac{1}{z}}\right) &= 1-\alpha\left(-\left(\delta+\frac{1}{z}\right)\right)\\ &= 1-\alpha\left(-\frac{1}{z}\right)=\alpha(z).\end{aligned}$$

Thus, $\alpha(Bz)=\alpha(z)$. Next, we have

$$\begin{aligned}\beta(A'z) &= \beta\left(z+\frac{\delta}{n}\right)\\ &= \alpha\left(n\left(z+\frac{\delta}{n}\right)\right)\\ &= \alpha(nz+\delta)\\ &= \alpha(nz)=\beta(z)\end{aligned}$$

and

$$\begin{aligned}\beta(B'z) &= \beta\left(\frac{z}{n\delta z+1}\right)\\ &= \alpha\left(n\left(\frac{z}{n\delta z+1}\right)\right)\\ &= \alpha\left(\frac{1}{\delta+\frac{1}{nz}}\right)\\ &= 1-\alpha\left(-\left(\delta+\frac{1}{nz}\right)\right)\\ &= 1-\alpha\left(-\frac{1}{nz}\right)\\ &= \alpha(nz)=\beta(z).\end{aligned}$$

□

Conclusion: In this study, we have investigated the automorphicity of the moduli α and β of the generalized modular equation (1). The generators of the Hecke subgroups K associated with the generalized modular equation have been determined. It has been proven that the modulus α is the automorphic function on $K=\langle A,B\rangle$, where

$$A=\begin{pmatrix}1 & \delta\\ 0 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 0\\ \delta & 1\end{pmatrix}$$

and the modulus β is the automorphic function on $K'=\langle A',B'\rangle$, where

$$A'=\begin{pmatrix}1 & \frac{\delta}{n}\\ 0 & 1\end{pmatrix} \text{ and } B'=\begin{pmatrix}1 & 0\\ n\delta & 1\end{pmatrix}$$

for $\delta=2\cos\frac{(1-2s)\pi}{2}$, $s\in\left(0,\frac{1}{2}\right]$.

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